INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 9, 1113–1119 (1989)

A VELOCITY FORMULATION FOR FLOW PAST A SYMMETRIC PROFILE WITH A WAKE

S. M. RÄSÄNEN

Institute of Mechanics, Helsinki University of Technology, SF-02150 Espoo, Finland

SUMMARY

A model having velocity components as basic unknowns is presented for calculation of two-dimensional flow past a symmetric profile with a wake in a channel. A modified least squares functional is used for the finite element solution of velocities. The determination of the position of the free streamline is treated as an optimum design problem. The concepts of cost function, geometry parameter and sensitivity derivative are employed. Numerical results are compared with published results obtained with streamfunction formulations.

KEY WORDS Finite element Free streamlines Wakes

INTRODUCTION

Ideal (incompressible, irrotational) fluid flow is often used as an approximation when modelling flow past profiles with a wake. This approximation is fairly well applicable in the region outside the wake.[1] The boundary between the outside flow and the wake is characterized by a constant pressure, and accordingly by a constant velocity. The boundary is called the free streamline. The position of the free streamline is unknown in the beginning, which is the main source of difficulties in solving wake flows.

Problems of flow past profiles have traditionally been formulated and solved in terms of the streamfunction (as in fact have many other ideal fluid flow problems). The solution procedures[2–6] are typically such that the problem is transformed from the physical plane to a convenient space where it is formulated anew and where a numerical method is applied. The use of variable finite elements connected with the minimization of a functional over a variable region has also been suggested.[7]

In this paper a direct method is presented—'direct' meaning that velocity components are used as basic unknowns. The approach leads to a notably simple formulation. In addition, it is often the velocities that we are really interested in, for instance if the forces acting on the profile were to be calculated. From the streamfunction the velocities can be obtained only after accuracy-decreasing numerical derivations.

FORMULATION OF THE PROBLEM

The geometry of the problem is shown in Figure 1. Owing to the symmetry, it is only necessary to consider the region ABCDEF.

0271–2091/89/091113–07$05.00 *Received 16 June 1988*

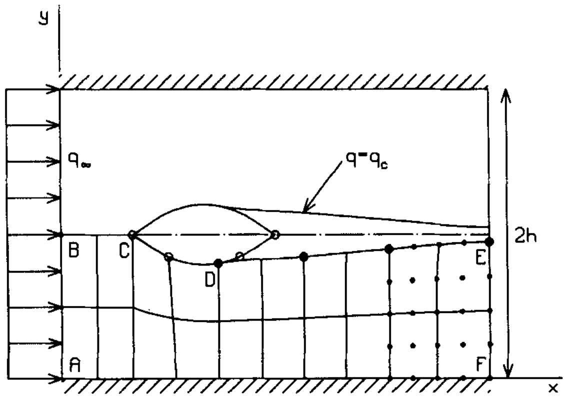

Figure 1. Geometry of the problem: ●, geometry point on the free streamline; ○, geometry point on the surface of the profile; •, node

The velocity components u and v in the x and y directions satisfy the continuity equation and the irrotationality condition:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad \text{in ABCDEF,} \tag{1}$$

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0 \quad \text{in ABCDEF.} \tag{2}$$

Far from the profile the flow in the channel is assumed to be uniform. The boundary conditions are

$$u=q_\infty \qquad \text{on AB,} \tag{3a}$$

$$v=0 \qquad \text{on AB, BC, EF, FA,} \tag{3b}$$

$$q_n=0 \qquad \text{on CD, DE,} \tag{3c}$$

$$q=q_c \qquad \text{on DE,} \tag{3d}$$

where q_∞ denotes the far field magnitude of the velocity, q_n is the velocity component perpendicular to the boundary and q_c is the constant value of the velocity on the free streamline.

It should be emphasized that the position of the free steamline DE is not known, including the position of the separation point D, and that there are two conditions to be satisfied on this boundary.

SOLUTION PROCEDURE

Description of the free streamline

The free streamline is described with a smooth curve that passes through so called geometry points (see Figure 1). Between two successive geometry points the position of the streamline is given by the mapping

$$\mathbf{r}(\xi)=N_1(\xi)\mathbf{r}_1+N_2(\xi)\mathbf{r}'_1+N_3(\xi)\mathbf{r}_2+N_4(\xi)\mathbf{r}'_2, \tag{4}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the position vectors of the two geometry points and $\mathbf{r}'_1$ and $\mathbf{r}'_2$ are the corresponding derivatives with respect to the parameter ξ. The values of ξ range from 0 to 1. The functions N_i are the Hermitian cubic polynomials

$$\begin{aligned} N_1(\xi) &= 1-3\xi^2+2\xi^3, \qquad & N_2(\xi) &= \xi-2\xi^2+\xi^3, \\ N_3(\xi) &= 3\xi^2-2\xi^3, \qquad & N_4(\xi) &= -\xi^2+\xi^3. \end{aligned} \tag{5}$$

The positions of the geometry points and the tangent vector directions are calculated in the course of the procedure, as explained later. Before equation (4) can be applied, the tangent vectors $\mathbf{r}'_1$ and $\mathbf{r}'_2$ must also be known. They are calculated according to the equations[8]

$$\mathbf{r}'_1 = \frac{d}{\mathbf{t}_1 \cdot \mathbf{e}}\mathbf{t}_1, \qquad \mathbf{r}'_2 = \frac{d}{\mathbf{t}_2 \cdot \mathbf{e}}\mathbf{t}_2, \tag{6}$$

where $\mathbf{t}_1$ and $\mathbf{t}_2$ are the unit tangent vectors at the geometry points, $\mathbf{e}$ is the unit vector in the direction of the chord $\mathbf{r} = (1-\xi)\mathbf{r}_1 + \xi\mathbf{r}_2$ connecting the geometry points, and d is the chord length. The geometric significance of expressions (6) is that with the same value of ξ the line segment joining a point on the curve and a point on the corresponding chord is perpendicular to the chord.

The shape of the profile is discretized in just the same way as the free streamline. The profile geometry of course remains fixed during the calculations, whereas the position of the streamline changes.

The finite element mesh is generated to follow the free streamline and the surface of the profile as shown schematically in Figure 1.

Velocities

A modified least squares functional

$$\Pi(u, v; \lambda_1, \lambda_2) = \frac{1}{2}\int_A \left[\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)^2\right] \mathrm{d}A + \int_A \left[\lambda_1\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) + \lambda_2\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\right] \mathrm{d}A \tag{7}$$

forms the basis of the finite element discretization. The velocity components are approximated with a C^0-continuous representation (nine-noded isoparametric Lagrangean quadrilaterals) and the Lagrange multipliers λ_1 and λ_2 with a C^{-1}-continuous representation (elementwise constant values). The standard least squares functional does not give accurate enough results for reasonable meshes. The addition of the Lagrange multiplier terms improves the accuracy, which can be demonstrated for instance by studying the overall mass conservation.[9]

Stationarity of functional (7) gives rise to a system of equations

$$\mathbf{K}\mathbf{a} = \mathbf{p} \tag{8}$$

for solving the unknown velocity components at the nodes of the element mesh together with the Lagrange multipliers.

Numerical experiments indicate that the following boundary conditions, for instance, are sufficient to solve equations (8): the direction of the velocity is given at all boundary nodes and the magnitude of the velocity is given at one node. Thus the velocity formulation allows the boundary conditions to be modelled realistically, which is of great use in many fluid flow problems, for instance in open channel flow.[10]

The actual velocity boundary conditions used here will be specified in the example section.

Optimum design approach

So called design parameters are defined with respect to a current geometry. Referring to the shape of the free streamline, the design parameters are selected to be a change in the position of the geometry point in the direction perpendicular to the streamline and a change in the direction angle of the tangent to the streamline at the geometry point. Thus there are two design parameters connected with one geometry point. The design parameters are denoted by b_j.

The selection of the design parameters must be modified at the geometry point coinciding with the separation point. To maintain the point in contact with the profile, it is required as a first approximation that in a change

$$\mathbf{r}(\xi+\Delta\xi)\approx\mathbf{r}(\xi)+|\mathbf{r}'(\xi)|\Delta\xi\mathbf{t}=\mathbf{r}(\xi)+\Delta b\mathbf{t}, \tag{9}$$

where $\mathbf{r}$ is the position vector of the geometry point, ξ is its current co-ordinate on the boundary curve of the profile and $\mathbf{t}$ is the unit tangent vector. Equation (9) gives the change in the ξ-co-ordinate $\Delta\xi=\Delta b/|\mathbf{r}'(\xi)|$ when a change of magnitude Δb is made in the position of the geometry point. At the separation point the tangent vector direction is not an independent geometry parameter. The streamline is tangential to the surface of the profile and thus the tangent direction is determined by the position of the point.

The determination of the position of the free streamline is based on minimizing the cost function

$$W=\tfrac{1}{2}\int_S(q-q_c)^2\mathrm{d}s\approx\tfrac{1}{2}\sum_i w_i(q_i-q_c)^2, \tag{10}$$

which requires boundary condition (3d) to be satisfied in a least squares sense. The integration is over the free streamline. Because the integral must be calculated numerically, it is replaced by the given pointwise counterpart where the summation is over the integration points on the streamline; w_i denotes the integration weight.

Changes in the design parameters lead to changes in velocities. The new value of the velocity at an integration point due to changes in the design parameters is

$$q_i^{\text{new}}\approx q_i^{\text{old}}+\sum_j\frac{\partial q_i}{\partial b_j}\Delta b_j=q_i^{\text{old}}+\sum_j c_{ij}\Delta b_j, \tag{11}$$

where the summation is over the number of design parameters. Equation (11) defines new quantities c_{ij}, called sensitivities.

Expression (11) is substituted into cost function (10) which is then minimized with respect to the changes Δb_j. The minimizing conditions

$$\partial W/\partial\Delta b_j=0,\quad j=1,2,\ldots, \tag{12}$$

lead to a linear system of equations for the unknowns Δb_j.

The sensitivities are calculated by making a small predefined change in every design parameter in turn and by employing the approximate relation

$$c_{ij}\approx\Delta q_i/\Delta b_j. \tag{13}$$

The whole element mesh is generated again after a change Δb_j. Changes in velocities are calculated from the approximate equation

$$\mathbf{K}\Delta\mathbf{a}=\Delta\mathbf{p}-\Delta\mathbf{K}\mathbf{a} \tag{14}$$

derived by differentiating equation (8) with respect to a design parameter. When solving equation

(14) for the changes $\Delta \mathbf{a}$, use is made of the fact that the matrix $\mathbf{K}$ has already been triangularized in connection with equation (8).

The above described procedure for calculating the changes Δb_j is repeated until convergence is achieved.

NUMERICAL EXAMPLES

Three-point Gaussian quadrature per element edge has been employed in connection with cost function (10). The predefined changes Δb_j^{pre} of the design parameters needed for calculating the sensitivities have been selected so that $\Delta b_j^{\text{pre}}/4h = 10^{-2}$ for a change in the position of the geometry point (h is half of the width of the channel), and the change in the direction angle of the tangent is adjusted to produce roughly the same maximum change in the position of the streamline. The convergence criterion has been $\max_j |\Delta b_j|/\Delta b_j^{\text{pre}} < 10^{-2}$.

The numerical examples to be considered have been selected mainly because they allow comparisons with previously published results obtained with streamfunction models.

Circular profile

As a first example, flow past a circular profile is considered (Figures 2 and 3). The radius of the circle is $h/4$. Velocities q_∞ and q_c are both assumed to be known. As boundary conditions to equations (8), the nodal values of the velocity components are given according to equations (3a), (3b) and (3c).

Figure 2 shows the free streamline where $q_c = q_\infty$. The result is obtained in twelve iterations from the assumed streamline position also shown in the figure. When solving the changes from the system of equations (12), the tangent vector direction at the outflow section is fixed to be horizontal.

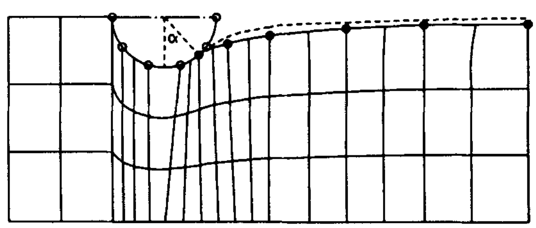

Figure 2. Flow past a circular cylinder, $q_c = q_\infty$, 6 geometry points, 30 integration points, 54 elements: ---, assumed streamline; —, result

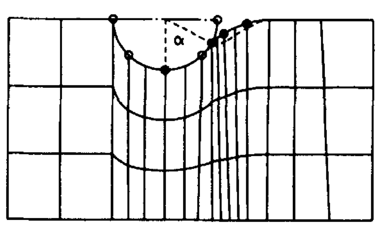

Figure 3. Flow past a circular cylinder, $q_c = 0{\cdot}67\ q_\infty$, 3 geometry points, 12 integration points, 48 elements: ---, assumed streamline; —, result

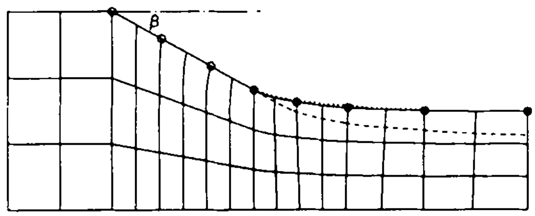

Figure 4. Flow past a wedge, 5 geometry points, 24 integration points, 48 elements: ---, assumed streamline; —, result; . . . . , Dormiani *et al.*[6]

The angle α at the separation point (defined in Figure 2) is found to be 41·8°. This can be compared with $\alpha = 42{\cdot}8°$ obtained by Dormiani and Bruch[4] with a streamfunction method.

Figure 3 shows the result obtained with $q_c = 0{\cdot}67\ q_\infty$. The Bernoulli equation $p + (\frac{1}{2})\rho q^2 =$ constant (ρ is the density) shows that if the pressure is required to have a minimum in the wake, then q_c should be the maximum velocity in the flow. However, wakes with $q_c < q_\infty$ are frequently treated in the literature—starting with Southwell and Vaisey[11]—and they can be thought of as a result from some kind of artificial blowing or ventilation.[1]

When $q_c < q_\infty$, the wake is finite and terminates with a cusp. The point where the streamline reaches the channel centreline is unknown. Therefore only the three geometry points marked in Figure 3 are employed and the integration in cost function (10) is accordingly over four elements only. Seven iterations were needed to obtain the result. At the separation point $\alpha = 62{\cdot}1°$. Dormiani and Bruch[4] give the value 60·8°.

In the velocity formulation the value of the volume flow Q is exact due to boundary condition (3a). The streamfunction models give only an approximate volume flow, for example $Q/Q_{exact} = 0{\cdot}958$ in the test case with $q_c = 0{\cdot}67\ q_\infty$.[4]

Wedge

The model is applicable also when the position of the separation point and the velocity at the free streamline are given, but the inflow velocity is assumed to be unknown. The wedge of Figure 4 is a profile where the geometry determines the separation point. The half-angle β of the wedge is 30° and the side length is 0·7875 h.

The velocity boundary conditions must be modified to correspond to the assumptions. Condition (3a) is dropped and the given value of q_c is used at the separation point.

Two geometry parameters are now fixed: the position of the separation point and the tangent vector direction at the outflow section. The tangent is again set to be horizontal.

The result shown in Figure 4 has been obtained in seven iterations. Also drawn in the figure is the result given by Dormiani *et al.*,[6] reproduced from a figure in their article. Their value of q_∞ is 0·501 q_c, while the present calculations give the average $q_\infty = Q/h = 0{\cdot}492\ q_c$.

It may be emphasized that in the velocity formulation the result, including the value of q_∞, is obtained after performing the described iterative procedure just once. In the streamfunction models[6,7] iterations must be performed with several assumed values of some of the unknowns (depending on the model), and the final result is selected according to some criterion, usually based on physical considerations.

CONCLUSION

In view of the numerical results obtained, the velocity formulation provides an efficient and conceptually straightforward alternative to the streamfunction methods for solving flows past profiles.

ACKNOWLEDGEMENT

The author wishes to thank Associate Professor E.-M. Salonen for his valuable comments on the manuscript.

REFERENCES

1. T. Y.-T. Wu, 'Inviscid cavity and wake flows', in M. Holt (ed.), *Basic Developments in Fluid Dynamics, Vol. 2*, Academic Press, New York, 1968, pp. 1–116.
2. J. F. Bourgat and G. Duvaut, 'Numerical analysis of flow with or without wake past a symmetric two-dimensional profile without incidence', *Int. j. numer. methods fluids*, **11**, 975–993 (1977).
3. R. T. Davis, 'Numerical methods for coordinate generation based on a mapping technique', in *Computational Methods for Turbulent, Transonic, and Viscous Flows*, Hemisphere, Washington, 1983, pp. 1–44.
4. M. Dormiani and J. C. Bruch, 'Numerical results for flow past a symmetric two-dimensional profile with a wake in a channel', in *Numerical Methods for Non-Linear Problems, Vol. 2*, Pineridge Press, Swansea, 1984, pp. 688–699.
5. J. C. Bruch and M. Dormiani, 'Free boundary fluid flow problems solved numerically in a hodograph plane using a fixed domain method', in *Numerical Methods for Non-Linear Problems, Vol. 3*, Pineridge Press, Swansea, 1986, pp. 693–726.
6. M. Dormiani, J. C. Bruch and J. M. Sloss, 'Flow past symmetric convex profiles with open wakes', *Int. j. numer. methods fluids*, **7**, 1301–1314 (1987).
7. J. M. Aitchison, 'The numerical solution of planar and axisymmetric cavitational flow problems', *Comput. Fluids*, **12**, 55–65 (1984).
8. S. M. Räsänen and E.-M. Salonen, 'A model for understanding free surface flow', in *Numerical Methods for Non-Linear Problems, Vol. 2*, Pineridge Press, Swansea, 1984, pp. 849–860.
9. E-M. Salonen, P. Lehtonen and A. Pramila, 'Further u, v-formulations for ideal fluid flow', in *Proc. 2nd Int. Conf. on Numerical Methods in Laminar and Turbulent Flow*, Pineridge Press, Swansea, 1981, pp. 1185–1196.
10. S. M. Räsänen and E-M. Salonen, 'A u, v-formulation for free surface flow', in *Proc. 4th Int. Conf. on Numerical Methods in Laminar and Turbulent Flow, Part 2*, Pineridge Press, Swansea, 1985, pp. 1393–1401.
11. R. V. Southwell and G. Vaisey, 'Relaxation methods applied to engineering problems XII. Fluid motions characterized by 'free' stream-lines', *Phil. Trans. R. Soc.*, **A240**, 117–161 (1946).